PENROSE VOTING SYSTEM AND OPTIMAL QUOTA*

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Systems of indirect voting based on the principle of qualified majority can be analysed using the methods of game theory. In particular, this applies to the voting system in the Council of the European Union, which was recently a subject of a vivid political discussion. The *a priori* voting power of a voter measures his potential influence over the decisions of the voting body under a given decision rule. We investigate a system based on the law of Penrose, in which each representative in the voting body receives the number of votes (the voting weight) proportional to the square root of the population he or she represents. Here we demonstrate that for a generic distribution of the population there exists an optimal quota for which the voting power of any state is proportional to its weight. The optimal quota is shown to decrease with the number of voting countries.

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Voting rules implemented by various political or economical bodies may be studied with the help of the tools developed for many decades in game theory [6, 14, 17, 38]. We are going to analyse a special case of indirect voting: each citizen of a given country elects a representative, who will cast a ballot in the voting body on behalf of his electors. The decisions of such a

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body are taken if certain fixed conditions characterising qualified majority (the winning coalition) are fulfilled. For instance, according to the agreement reached in Brussels in June 2004 and signed in Rome in October 2004, the Council of Ministers of the European Union (EU) acting on a proposal from the Commission or from the Union Minister for Foreign Affairs takes its decisions if two criteria are simultaneously satisfied: (a) at least 55% of members of the Council, comprising at least fifteen of them vote 'yes', and (b) these members represent Member States comprising at least 65% of the total population of the Union. Additionally: (c) a blocking minority must include at least four Council members, failing which the qualified majority shall be deemed attained. The same rules apply to the European Council when it is acting by a qualified majority (The Treaty Establishing a Constitution for Europe, 2004, see [48]).

A mathematical theory of indirect voting was initiated after World War II by British psychiatrist and mathematician Lionel S. Penrose (1946) in the context of a hypothetical distribution of votes in the UN General Assembly [38]. He introduced the concept of *a priori* voting power, a quantity measuring the ability of a participant X of the voting body to influence the decisions taken. In 1965 a similar analysis was independently carried out by American attorney John F. Banzhaf III [5]. The voting power is proportional to the probability that a vote cast by X in a hypothetical ballot will be decisive: a winning coalition would fail to satisfy the qualified majority condition without X or a losing coalition would start to satisfy it with X. If we assume that all potential coalitions are equally probable, then the voting power may be expressed by the *Penrose–Banzhaf index* (*PBI*) [14,17], called also the Banzhaf index. For convenience one often normalises the PBIs in such a way that their sum is equal to unity. The relative voting power should be distinguished from the voting weight: a shareholder with 51% of stocks of a company has only 51% of all votes at the shareholders assembly, but he takes 100% of the voting power if the assembly votes by a simple majority rule. Note that this approach is purely *normative*, not descriptive: we are interested in the *a priori* voting power arising from the voting procedure itself. The actual voting power depends on the polarisation of opinion in the voting body and changes from voting to voting [19–21, 37].

To compute the PBIs of M participants of a voting system which follows a given set of rules one needs to consider all possible 2^M coalitions to check which of them satisfies the qualified majority condition, and to count those for which the voice of a given participant is decisive. In the case of the EU consisting of 25 (or in the near future 27) states, there are more than 33.5 (or, respectively, 134) millions of possible coalitions. A game-theoretical analysis of the rules of voting in the European Council performed along those lines shows [4, 5, 11, 13, 39] that the double majority system laid down in 2003 by

the European Convention attributes a much smaller relative voting power to Spain and Poland than the earlier system accepted in the Treaty of Nice in 2001. In this way we obtain a mathematical explanation of the political fact that these two countries were the main opponents of the proposed changes to the voting rules [1, 5, 11].

To describe an algorithm of computing the PBIs assume that ω is the number of winning coalitions, in the sense that they satisfy the qualified majority rule adopted. There exist 2^{M-1} different coalitions in which a given country can take part. Let ω_x denote the number of winning coalitions that include the country x. Assuming that all 2^M coalitions are equally likely we can compute the probability that a vote cast by x is decisive. This happens, if x is a *critical voter* in a coalition, *i.e.*, the winning coalition (with x) ceases to fulfil the majority requirements without x. The number of these cases is: $\eta_x = \omega_x - (\omega - \omega_x) = 2\omega_x - \omega$. The *absolute Penrose–Banzhaf index* is equal to the probability that x is critical: $B_x = \eta_x / 2^{M-1}$. To compare these indices for decision bodies consisting of different number of players, it is convenient to define the *normalised Penrose–Banzhaf index*: $\beta_x = \left(\sum_{x=1}^{M} \eta_x\right)^{-1} \eta_x$. Penrose mentioned in 1946 that in this model the probability p_x that the country x is on the 'winning' side reads:

$$p_x = \frac{\omega_x + \left(2^{M-1} - (\omega - \omega_x)\right)}{2^M} = \frac{1 + B_x}{2}\,,$$

and so it is a function of the absolute Banzhaf index.

Which voting system is fairer and more accurate? A partial answer to this question was already given by Penrose [38], who deliberated principles of an ideal representative voting system, in which every citizen of every country has the same potential voting power. First consider direct elections of the government (which nominates the minister voting on behalf of the entire country in the European Council) in a state with population N. It is easy to imagine that an average German citizen has smaller influence on the election of his government than, for example, a citizen of the neighbouring Luxembourg. Making use of the Bernoulli scheme and the Stirling approximation of the binomials, Penrose proved that in such elections the voting power of a single citizen decays as $1/\sqrt{N}$, given that the votes of citizens are uncorrelated. Thus, the system of indirect voting applied to the European Council would be representative in this sense, if the voting power of each country behaved *proportionally to* $\sqrt{N}$, so that both factors cancelled out. (This has a direct physical analogy with the random walk of a diffusing particle [43].) This statement, known in the literature under the name of the *square root law of Penrose* [14], was independently proposed in the EU context by Laruelle and Widgrén [27], see [26] for an earlier version. Since then potential voting systems in the EU Council of Ministers that obey Penrose's square root

law have been analysed by many authors [3,15,16,22,24,25,34,39,46,47,49]. (Other arguments for the optimality of the square root formula can be found in [7, 8, 32, 33, 40].) Such voting procedures has been also used in practice in other international institutions, for example, in the Canadian Geoscience Council, the International Federation of Operational Research Societies, the International Genetics Federation, the International Mycological Association, and the World Federalist Movement. However, it is not clear in general how to solve directly the *inverse problem*, *i.e.*, how to allocate weights and how to define qualified majority rules to obtain required distribution of power [27, 28, 31, 36, 45, 50].

To this end we proposed [42, 51] a voting system exploiting a single criterion: the voting weight of each Member State is allocated proportionally to the square root of its population, the decision of the Council being taken if the sum of weights exceeds a certain *quota* (threshold) R. Taking the populations N_x $(x = 1, \ldots, 25)$ of all 25 EU member states as of 1 January 2003 [1] we analysed their voting powers in this system as functions of the quota R. Fig. 1 shows the ratio of the normalised PBIs $\beta_x(R)$ to the voting weights proportional to $\sqrt{N_x}$ for five exemplary states. Interestingly, all 25 curves (for transparency only 5 are plotted here) cross approximately at a single point for a critical quota $R^{25}_{\rm opt} = 62\%$. Fig. 2 illustrates the dependence of the square root of the sum of square residuals σ between the normalised PBIs and voting weights on the value of the threshold R, where

$$\sigma^2 = \sum_{x=1}^{M} \left(\beta_x(R) - \sqrt{N_x} \Big/ \sum_{y=1}^{M} \sqrt{N_y} \right)^2 .$$

Since the minimum value of this function attained for $R^{25}_{\rm opt}$ is very small (approximately 0.0003), we are able to work out the optimal value for the threshold for which both the voting powers and weights coincide. For this very choice of the quota the computed voting power of each country is practically equal to the attributed voting weight, and so it is proportional to the square root of the population. Hence the Penrose law is almost exactly fulfilled, and the potential influence of every citizen of each Member State on the decisions taken in the Council is the same. Such a voting system is not only representative but also *transparent*: the voting powers are proportional to the voting weights. Furthermore, the system is simple (one criterion only), easily extendible and objective: it does not favour nor handicap any European country. It has been christened by the media as the 'Jagiellonian Compromise'.

[1] Data from *EUROSTAT*: First results of the demographic data collection for 2003 in Europe. Statistics in focus. Population and social conditions 2004; 13; 1–7.

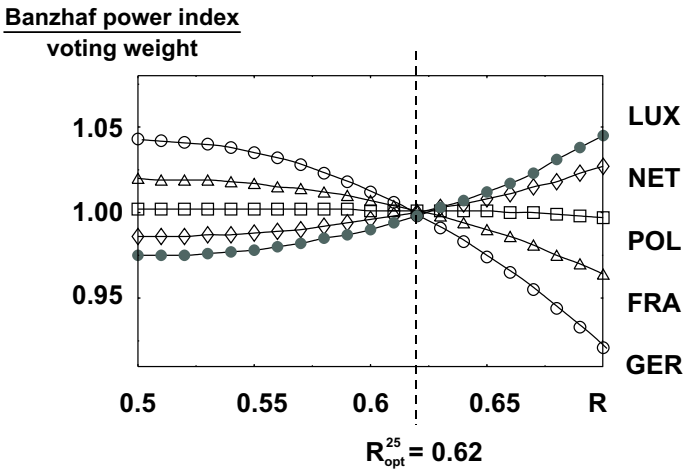

Fig. 1. Ratio of voting power to voting weight as a function of the quota for five exemplary states of EU-25 (Luxembourg, the Netherlands, Poland, France, and Germany); all functions cross near the critical point $R^{25}_{\rm opt} = 62\%$.

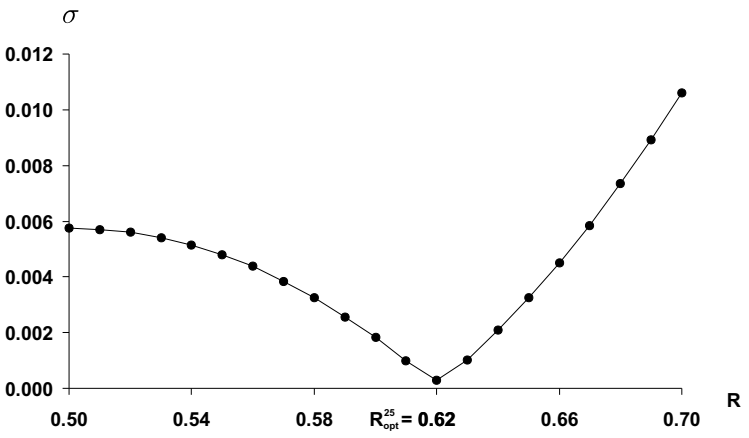

Fig. 2. The cumulative residual σ between the voting weight and power for all EU-25 countries as dependent on the value of the threshold R.

The main result of our work consists in the statement that the above method is not restricted to the actual distribution of population in European countries. Performing similar investigations for a hundred randomly

chosen populations of fictitious 'Unions' containing M states each, for every realisation we have found a critical quota $R_{\rm opt}$ at which the voting powers of all 'members' approximately coincide with the weights allocated. Thus, our method provides in many cases a surprisingly simple solution of the inverse problem. The value of the critical quota is realisation dependent, but for a fixed M the fluctuations are small. Moreover, the critical quota decreases with the size of the 'Union', which is rather intuitive: the more countries, the more difficult it becomes to construct a winning coalition. For instance, for the Council of Ministers of EU-27 (including also Romania and Bulgaria) the optimal quota $R^{27}_{\rm opt} \simeq 61.4\%$, see Table I.

In the limiting case as $M \to \infty$ the critical quota seems to tend to 50%, consistently with the so-called Penrose limit theorem [30,31]. The existence of the optimal quota was confirmed in a recent study by Chang, Chua, and Machover [12] who, however, used different measure on the set of distributions of population. Table II shows the value of the mean critical quota as a function of the number M of members of the voting body. The data were obtained by averaging over the sample of 50 realizations of random populations generated with respect to the statistical measure, *i.e.*, the symmetric Dirichlet distribution with Jeffreys' priors [41] with the density given by

$$P\left(x_{1,\ldots,}x_M\right) = C_M \left(x_1 \cdot \ldots \cdot x_M\right)^{-1/2}$$

for $x_i \geq 0$, $\sum_{i=1}^{M} x_i = 1$, where the normalisation constant is expressed by the Euler gamma function, $C_M := \Gamma\left(M/2\right) \pi^{-M/2}$. This measure on the simplex of probability distributions has been selected since it is induced by the Fisher–Mahalanobis–Battacharyya–Rao Riemannian metric on this set, which in turn is distinguished by being invariant under reparametrisation [2].

The above result has a simple practical meaning: for a given number of states M, choosing weights proportional to the square root of the population and the quota in the close vicinity of $R^M_{\rm opt}$ we assure that the system is (according to the Penrose law) nearly optimally representative, since the voting power of each country becomes proportional to the square root of its population, and so the voting power of every citizen of each state is nearly the same.

The representative voting system based on the square root law of Penrose and the appropriate choice of optimal quota may be used as a reference point to analyse the rules established by politicians. Fig. 3 presents a comparison of the voting power (measured by the PBI) of EU members according to the system accepted in Brussels in June 2004 (applied to EU-27, including also Romania and Bulgaria) and according to the Penrose solution with the optimal quota $R^{27}_{\rm opt} = 61.4\%$, see [9,10,18,23,44] for similar analyses. The double majority rule is beneficial to the largest countries (Germany, France,

TABLE I

Comparison of voting power of EU-27 member states in the system of the European Constitution and in the proposed solution ('Jagiellonian Compromise') based on the Penrose law with the threshold $R_{\rm opt}^{27} = 61.4\%$.

Member state	Population (in millions)	Voting power (Constitution)	Voting weight (Penrose)	Voting power (Penrose)
Germany	82.54	11.87	9.55	9.54
France	59.64	8.74	8.11	8.12
United Kingdom	59.33	8.69	8.09	8.10
Italy	57.32	8.44	7.95	7.96
Spain	41.55	6.37	6.78	6.79
Poland	38.22	5.89	6.49	6.50
Romania	21.77	4.22	4.91	4.91
Netherlands	16.19	3.51	4.22	4.22
Greece	11.01	2.88	3.49	3.49
Portugal	10.41	2.80	3.39	3.39
Belgium	10.36	2.80	3.38	3.38
Czech Republic	10.20	2.78	3.35	3.35
Hungary	10.14	2.77	3.34	3.34
Sweden	8.94	2.63	3.14	3.14
Austria	8.08	2.52	2.98	2.98
Bulgaria	7.85	2.49	2.94	2.94
Denmark	5.38	2.19	2.44	2.44
Slovakia	5.38	2.19	2.44	2.44
Finland	5.21	2.17	2.39	2.39
Ireland	3.96	2.02	2.09	2.09
Lithuania	3.46	1.96	1.95	1.95
Latvia	2.33	1.82	1.61	1.61
Slovenia	2.00	1.78	1.48	1.48
Estonia	1.36	1.70	1.23	1.23
Cyprus	0.72	1.62	0.89	0.89
Luxembourg	0.45	1.59	0.70	0.70
Malta	0.40	1.58	0.66	0.66

TABLE II

Average optimal threshold $R_{\rm opt}^{M}$ as a function of the number of states M.

M	10	12	14	16	18	20	22	24	26
$R_{\rm opt}^{M}$	66.0%	65.8%	64.6%	64.4%	63.4%	63.1%	62.6%	62.0%	61.4%

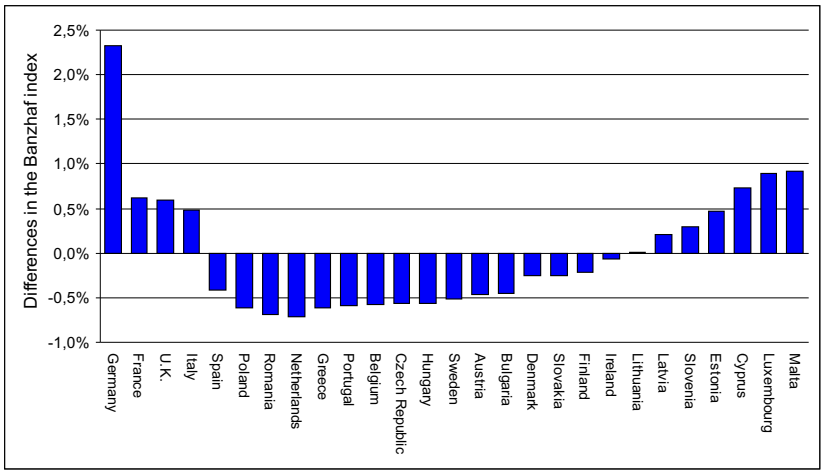

Fig. 3. Differences in voting power in the EU-27 Council between the system of the European Constitution and the proposed solution ('Jagiellonian Compromise') based on the Penrose law with $R^{27}_{\rm opt} = 61.4\%$. The member states are ordered according to their population.

the United Kingdom, and Italy), due to the 'per capita' criterion, and to the smallest countries (from Latvia to Malta), for which the condition 'per state' plays a key role. Since the largest and the smallest countries gain relative voting power, it is easy to see that this occurs at the expense of all the medium-sized countries (from Spain to Ireland), which from this point of view are handicapped by the Treaty Establishing a Constitution for Europe.

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